Paper Reference(s)

9801

Mathematics

Advanced Extension Award
(Trial Examination)

Summer 2001

Time: 3 hours

Materials required for examination	Items included with question papers
Answer Book (AB16) Graph Paper (ASG2) Mathematical Formulae	Nil

Candidates may use any calculator EXCEPT those with a facility for symbolic algebra, differentiation and/or integration. Thus candidates may NOT use calculators such as Texas TI89, TI92, Casio 9970G, Hewlett Packard HP48G.

Instructions to Candidates

Full marks may be obtained for answers to ALL questions in Section A and TWO questions in Section B.
In the boxes on the answer book, your centre number, candidate number, the paper title, the paper reference (9801), your surname, other names and signature.

Information for Candidates

A booklet 'Mathematical Formulae including Statistical Formulae and Tables' is provided.
This paper has 13 questions. Page 12 is blank.

Advice to Candidates

In calculations you are advised to show all the steps in your working, giving your answer at each stage.

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SECTION A

Answer ALL the questions in this section.

1. (a) By considering the series

$$1 + t + t^2 + t^3 + \ldots + t^n,$$

or otherwise, sum the series

$$1 + 2t + 3t^2 + 4t^3 + \ldots + nt^{n-1}$$

for $t \neq 1$.

(5)

(b) Hence find and simplify an expression for

$$1 + 2\times 3 + 3\times 3^2 + 4\times 3^3 + \ldots + 2001\times 3^{2000}.$$ **(1)**

(c) Write down expressions for the sums of both of the series in part (a) for the case where $t = 1$.

(1)

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2. Given that $S = \int_0^{\frac{\pi}{2}} e^{2x} \sin x \, dx$ and $C = \int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$,

(a) show that $S = 1 + 2C$,

(3)

(b) find the exact value of S.

(6)

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3. Find, to 1 decimal place, the values of θ that satisfy $0 \leqslant \theta \leqslant 360$ and

$$6 \cos 4\theta^\circ + 2 \cos^2 \theta^\circ = 1.$$

(12)

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4. The variable y is defined by

$$y = \ln(\sec^2 x + \text{cosec}^2 x) \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

A student was asked to prove that

$$\frac{dy}{dx} = -4\cot 2x.$$

The attempted proof was as follows:

$$\begin{aligned} y &= \ln(\sec^2 x + \text{cosec}^2 x) \\ &= \ln(\sec^2 x) + \ln(\text{cosec}^2 x) \\ &= 2\ln\sec x + 2\ln\text{cosec}\, x \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 2\tan x - 2\cot x \\ &= \frac{2(\sin^2 x - \cos^2 x)}{\sin x \cos x} \\ &= \frac{-2\cos 2x}{\frac{1}{2}\sin 2x} \\ &= -4\cot 2x \end{aligned}$$

(a) Identify the error in this attempt at a proof. **(1)**

(b) Give a correct version of the proof. **(5)**

(c) Find and simplify a relationship between p and q, where p and q are two functions of x, such that the student would obtain the correct answer when differentiating $\ln(p + q)$, with respect to x, by the above incorrect method. **(8)**

5. The function f is defined on the domain $[-2, 2]$ by:

$$f(x) = \begin{cases} -kx(2+x), & -2 \leqslant x < 0, \\ kx(2-x), & 0 \leqslant x \leqslant 2. \end{cases}$$

where k is a positive constant.

The function g is defined on the domain $[-2, 2]$ by $g(x) = (2.5)^2 - x^2$.

(a) Prove that there is a value of k such that the graph of f touches the graph of g.

(8)

(b) For this value of k sketch the graphs of the functions f and g on the same axes, stating clearly where the graphs touch.

(4)

(c) Find the exact area of the region bounded by the two graphs.

(6)

SECTION B

Attempt TWO questions in this Section.

PURE MATHEMATICS

6. Find the following sums:

(a) $\displaystyle\sum_{r=1}^{n} \mathrm{e}^r$,

(2)

(b) $\displaystyle\sum_{r=1}^{n} \ln\left(\frac{r+2}{r}\right)$,

(6)

(c) $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)}$,

(5)

(d) $\displaystyle\sum_{r=1}^{n} \binom{n}{r} \tan^{2r}\theta, \quad \theta \neq \frac{k\pi}{2}$, where k is an integer.

(5)

One of these sums is convergent as $n \longrightarrow \infty$.

(e) State which sum it is and give the value of the limit.

(2)

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7. (a) Find a complex number z such that $z^2 = 5 - 12\mathrm{i}$.

(5)

The complex numbers α and β are the solutions of the equation

$$x(x + 1 - 4\mathrm{i}) = 5 - \mathrm{i}$$

and $\arg(\beta) > \arg(\alpha) > 0$.

(b) Find α and β, giving your answers in the form $a + \mathrm{i}b$.

(7)

(c) On the same Argand diagram plot the points representing the complex numbers α, β, $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

(5)

(d) Show that α^4 is a real number and find the exact value of α^{2001}, leaving your answer in the form $k^n(a + \mathrm{i}b)$ where $k, n, a, b, \in \mathbb{R}$.

(3)

MECHANICS

8. (a) Two particles A and B have masses m and $2m$ respectively and are joined by a light inextensible string of length l. Initially B is at rest on a smooth horizontal surface when A, moving with speed u on the same surface, collides directly with B. After the collision A and B move in a straight line and the speed of A is $\frac{1}{4}u$. Air resistance should be ignored.

(i) Find the speed of B directly after the collision.

(2)

(ii) Find the time until the string becomes taut.

(2)

After the string becomes taut, both particles move with the same speed w.

(iii) Find w.

(2)

(b) **Figure 1**

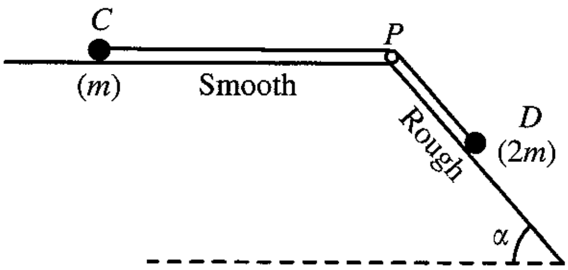

Figure 1 shows two particles C and D having masses m and $2m$ respectively and joined by a light inextensible string. Particle D is moving down a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The string passes over a smooth pulley P. Both particles have the same initial speed and particle C moves along a smooth horizontal surface whilst D moves down a line of greatest slope of the inclined plane. The motion of C and D is in a vertical plane and the coefficient of friction between D and the inclined plane is μ.

Find the acceleration of D and describe the motion of the particles in the period before C reaches the pulley in all the cases that arise according to the values of μ.

(14)

9. A girl is throwing screwed up pieces of paper into a large cylindrical bin of diameter 0.3 m. She projects them towards the bin with speed u at an angle α above the horizontal. The pieces of paper are modelled as particles that experience no air resistance.

(a) When the girl is sitting, the point of projection of the paper is the same height as the top of the bin and a horizontal distance 1.25 m from the nearest point of the bin. Given that $u = 4\text{ m s}^{-1}$ find, to the nearest 0.1 of a degree, the ranges of values of α for which the paper will land in the bin.

(9)

(b) When the girl is standing, the point of projection of the paper is 0.5 m above the level of the top of the bin and the horizontal distance between the point of projection and the nearest point on the top of the bin is 2.5 m. Given that $\alpha = 45°$ find, in m s^{-1} to 1 decimal place, the range of values of u that will ensure that the paper lands in the bin.

(9)

(c) Discuss briefly what the effect would be of using these values for u and α, but not making the modelling assumptions about the screwed up paper.

(2)

STATISTICS

10. A factory receives deliveries of a large number of a component, a proportion p of which are defective. Each delivery is tested for quality before being accepted. The quality assurance manager decides whether or not to accept the delivery on the basis of the following scheme.

A random sample of 10 components is examined and if no more than 1 of them is defective the delivery is accepted. If more than 2 defectives are found the whole delivery is rejected. If there are exactly 2 defective components in this first sample then a second sample of 10 components is examined. If there are no defectives in this second sample, the whole delivery is accepted; otherwise it is rejected.

(a) Show that the probability that the manager accepts the delivery is $q^9(q + 10p + 45p^2q^9)$ where $q = 1 - p$.

(5)

(b) Find an expression for the expected number of components sampled using the manager's scheme.

(5)

A new assistant starts working for the quality assurance manager and she decides to simplify the scheme. She examines a random sample of 15 components and accepts the delivery if there are no more than 2 defective components; otherwise the delivery is rejected.

(c) Find an expression for the probability that the assistant will accept a delivery.

(3)

The cost of testing each component sampled is £5, and the cost of rejecting a delivery is £1000 if $p = 0.05$ and £500 if $p = 0.10$.

(d) By considering the expected costs of sampling and rejecting deliveries in the cases where $p = 0.05$ and $p = 0.10$, explain which scheme you would recommend the factory to use in each of these cases.

(7)

11. Whenever Mr. Bumble gives a speech he makes hesitations at a rate of 1 every 2 minutes. The hesitations can occur at any point in his speech and are independent of one another.

(a) Find the probability that Mr. Bumble makes at least 3 hesitations in a 10-minute speech.

(2)

During a certain 10-minute speech by Mr. Bumble, Mr. Drowsy fell asleep. He remembers hearing at least 2 hesitations but does not know how many there were during the whole speech or when he fell asleep.

(b) Find the probability that there were at least 5 hesitations in the whole 10-minute speech.

(5)

Mr. Tally decides to count the number of complete minutes with no hesitations in a 10-minute speech by Mr. Bumble.

(c) Find the probability that there are no more than 7 complete minutes with no hesitations in Mr. Bumble's next 10-minute speech.

(6)

(d) Explain briefly the difference between your answers to parts (a) and (c).

(1)

Whenever Mr. Articulate delivers a speech there is a probability of $\frac{1}{2}$ that he makes at least one hesitation in the first minute. Thereafter he makes hesitations at a rate of 1 every 3 minutes.

(e) Find the probability that Mr. Articulate makes at least 3 hesitations in a 10-minute speech.

(6)

DECISION MATHEMATICS

12. A firm manufactures three types of woollen rug *A*, *B* and *C*. The rugs are woven, on machines, from three different colours of wool, I, II and III. The amount of wool (in kg), the machine time required (in hours) and the profit gained in producing each rug is shown in the table below.

	Wool I	Wool II	Wool III	Machine time	Profit (£)
Type *A*	1	3	1	2	5
Type *B*	4	2	2	4	6
Type *C*	3	3	1	2	4

The firm is currently supplied with 900 kg, 440 kg and 240 kg of wool I, II and III respectively each week. There is a total of 510 hours of machine time available each week. The weekly profit is to be maximised.

(a) Formulate this information as a linear programming problem. Reduce it, with explanation, to a 3-variable, 3-constraint problem and solve it using the Simplex method.

(10)

Due to improved efficiency, an extra 40 kg of wool can be supplied each week. The 40 kg may be in any one colour or any combination of colours (e.g. 20 kg of type I, 10 kg of type II and 10 kg of type III).

(b) State which wool type should not be increased and explain why.

(1)

(c) Prove that none of the pivot elements chosen in part (a) need changing.

(6)

(d) State the new optimal solution.

(3)

13. A printer (P) delivers a weekly newspaper in bundles of one hundred to three warehouses A, B and C. The warehouses then dispatch the bundles to four wholesalers D, E, F and G. The bundles are opened and individual retailers collect the papers from the wholesalers in the quantities they need.

The printer can deliver up to 31, 28 and 15 bundles to A, B and C respectively.

A can dispatch up to 20 bundles to D, B can dispatch up to 18 bundles to E and 11 to F, C can dispatch up to 11 bundles to G.

D, E, F and G estimate that the total number of papers required by the retailers they supply will not exceed 2500, 2200, 1000 and 2000 respectively.

(a) Draw a weighted, directed network to represent this situation.

(4)

(b) Verify that this system can supply a maximum of 59 bundles of newspapers from the printers to the retailers, and draw a diagram to show how this may be achieved.

(4)

The editor of the newspaper wishes to increase supply to the shop owners. Extra supply arcs are added to the network. The table below shows the extra arcs and the maximum number of bundles each can supply. All other arcs remain unchanged.

New arc	A to E	C to F	E to D	F to G
Number of bundles	20	10	5	8

(c) Using your answer to part (b) as the initial flow, and adding these new arcs to your network, use the labelling procedure to find the maximum number of papers that can be supplied. You must state your flow-augmenting routes and their value, and draw a diagram to show how your maximal flow may be achieved.

(12)

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